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2001 J. Phys. A: Math. Gen. 34 2637

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## Exact solutions of $n$ -coupled harmonic oscillators related to $Sp(2n, R)$ Lie algebra

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Received 19 October 2000, in final form 11 January 2001

### Abstract

Exact solutions of  $n$ -coupled harmonic oscillators related to the  $Sp(2n, R)$  Lie algebra are derived using an algebraic method. It is found that the energy spectrum of the system is determined by one-boson excitation energies built on a vector coherent state of  $Sp(2n, R) \supset U(n)$ .

PACS numbers: 0220Q, 0365F

Coupled harmonic oscillators are useful in describing many physical systems, such as molecular vibrations [1, 2], generalized coherent states in optics [3, 4], and so on. It is well known that the dynamical symmetry group for  $n$ -uncoupled harmonic oscillators is  $U(n)$ . A natural extension is to include 0 and  $\pm 2\hbar\omega$  shifts among different levels, which leads to the symplectic group  $Sp(2n, R)$ . The  $Sp(6, R)$  case was successfully used to manifest nuclear collective motion [5], which incorporated core excitations of both quadrupole and monopole type into the shell model foundation of the nuclear collective model and thus led to the possibility of full microscopic calculation of nuclear collective phenomena [6].

The  $Sp(2n, R)$  algebra consists of  $n(2n + 1)$  generators  $E_{ij}$ ,  $T_{ij}^{(+)} = T_{ji}^{(+)}$ , and  $T_{ij}^{(-)} = (T_{ji}^{(-)})^\dagger$ , with  $1 \leq i, j \leq n$ , which satisfy the following commutation relations:

$$\begin{aligned} [T_{ij}^{(\pm)}, T_{lm}^{(\pm)}] &= 0 & [T_{ij}^{(-)}, T_{lm}^{(+)}] &= \delta_{il}E_{mj} + \delta_{im}E_{lj} + \delta_{jl}E_{mi} + \delta_{jm}E_{li} \\ [T_{ij}^{(-)}, E_{lm}] &= \delta_{il}T_{jm}^{(-)} + \delta_{jl}T_{im}^{(-)} & [E_{ij}, E_{lm}] &= \delta_{jl}E_{im} - \delta_{im}E_{lj}. \end{aligned} \quad (1)$$

The generators  $\{E_{ij}\}$  with  $1 \leq i, j \leq n$  form the subalgebra  $U(n)$ .

An algebraic approach in some cases is a powerful procedure for solving energy eigenvalue problems. Some typical examples were shown in [7] and [8]. A lot of applications to the structure of nuclei can be found in [9]. In the following, we will outline an algebraic procedure for diagonalizing the Hamiltonian with a linear combination of all the generators of  $Sp(2n, R)$ , namely

$$\hat{H} = c + \sum_{ij} d_{ij} E_{ij} + \left( \sum_{ij} B_{ij} T_{ij}^{(+)} + \text{h.c.} \right) \quad (2)$$

where  $c, d_{ij} = d_{ji}^*$ , and  $B_{ij}$  are parameters of the system, and the generators of  $Sp(2n, R)$  are realized by  $n$ -boson operators with

$$\begin{aligned} E_{ij} &= a_i^\dagger a_j \\ T_{ij}^{(+)} &= a_i^\dagger a_j^\dagger & T_{ij}^{(-)} &= (T_{ji}^{(+)})^\dagger = a_j a_i \end{aligned} \quad (3)$$

for  $1 \leq i, j \leq n$ . In (3),  $a_j^\dagger$  and  $a_j$  ( $j = 1, 2, \dots, n$ ) are boson creation and annihilation operators. A special form of (2) with

$$\begin{aligned} c &= \frac{\hbar}{2} \sum_{i=1}^n \omega_i & d_{jj} &= \hbar \omega_j & \text{for } j &= 1, 2, \dots, n \\ d_{ij} &= d_{ji} = \frac{\hbar}{2} \lambda_{ij}^x (\sqrt{m_i m_j \omega_i \omega_j})^{-1} + \frac{\hbar}{2} \lambda_{ij}^p \sqrt{m_i m_j \omega_i \omega_j} \\ B_{ij} &= B_{ji}^* = \frac{\hbar}{4} \lambda_{ij}^x (\sqrt{m_i m_j \omega_i \omega_j})^{-1} - \frac{\hbar}{4} \lambda_{ij}^p \sqrt{m_i m_j \omega_i \omega_j} \\ B_{jj} &= 0 & \text{for } j &= 0, 1, \dots, n \end{aligned} \quad (4)$$

describes  $n$ -harmonic oscillators with both momentum and coordinate couplings, of which the Hamiltonian can also be written as

$$\hat{H} = \sum_{i=1}^n \left( \frac{1}{2m_i} p_i^2 + \frac{1}{2} m_i \omega_i^2 x_i^2 \right) + \sum_{i \neq j} (\lambda_{ij}^p p_i p_j + \lambda_{ij}^x x_i x_j). \quad (5)$$

In this case, the boson operators used in (3) should be expressed as

$$a_j^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{m_j \omega_j}{\hbar}} x_j + i \frac{1}{\sqrt{m_j \omega_j \hbar}} p_j \right) \quad a_j = (a_j^\dagger)^\dagger \quad (6)$$

which was used to describe molecular vibrations in [1, 2]. Since the Lie algebra  $Sp(2n, R)$  is non-compact, any non-trivial unitary irreducible representation of  $Sp(2n, R)$  is infinite-dimensional. The  $n$ -boson sub-Hilbert space spans two infinite-dimensional unitary representations of  $Sp(2n, R)$ , in which one forms from states with all even number of bosons, and another from those with all odd number of bosons.

In order to diagonalize the Hamiltonian (2), one can first make a unitary transformation with

$$\begin{aligned} a_i^\dagger &= \sum_{\mu=1}^n \alpha_\mu^{(i)} b_\mu^\dagger \\ a_i &= (a_i^\dagger)^\dagger & \text{for } i &= 1, 2, \dots, n \end{aligned} \quad (7)$$

where  $\{b_\mu^\dagger\}$  is another set of boson creation operators. In (7) the parameters  $\alpha_\mu^{(i)}$  should satisfy

$$\begin{aligned} \sum_{\mu} \alpha_\mu^{(i)*} \alpha_\mu^{(j)} &= \delta_{ij} \\ \sum_{ij} d_{ij} \alpha_\mu^{(i)} \alpha_\nu^{(j)*} &= 0 & \text{for } \mu &\neq \nu. \end{aligned} \quad (8)$$

After transformation (7), the Hamiltonian (2) can be expressed as

$$\hat{H} = c + \sum_{\mu} \rho_\mu E_{\mu\mu}(b) + \left( \sum_{\mu\nu} A_{\mu\nu} T_{\mu\nu}^{(+)}(b) + \text{h.c.} \right) \quad (9)$$

where  $E_{\mu\mu}(b) = b_\mu^\dagger b_\mu$ ,  $T_{\mu\nu}^{(+)}(b) = b_\mu^\dagger b_\nu^\dagger$ , and

$$\begin{aligned} \rho_\mu &= \sum_{ij} d_{ij} \alpha_\mu^{(i)} \alpha_\mu^{(j)*} \\ A_{\mu\nu} &= \sum_{ij} B_{ij} \alpha_\mu^{(i)} \alpha_\nu^{(j)}. \end{aligned} \quad (10)$$

Since there are  $\pm 2\hbar\omega$  shifts among different harmonic oscillator levels, the ground state of (9) should be expanded in terms of power series of the operators  $T_{\mu\nu}^{(+)}$  acting on the boson vacuum. Similar to the Bethe ansatz, it can be shown that the primitive eigenstate of (9) can be written as a vector coherent state (VCS) of  $Sp(2n, R) \supset U(n)$  built on the lowest weight state of  $U(n)$ , of which the general theory was given in [10–12], namely, up to a normalization factor

$$|g\rangle = \Gamma_g|0\rangle = e^{\sum_{\mu\nu} z_{\mu\nu} T_{\mu\nu}^{(+)}(b)}|0\rangle \tag{11}$$

where  $z_{\mu\nu} = z_{\nu\mu}$  are  $c$ -numbers to be determined, and  $|0\rangle$  is the  $b$ -boson vacuum state. Using the Hausdorff–Campbell formula and the eigenequation

$$\hat{H}|g\rangle = E_g|g\rangle \tag{12}$$

one obtains an energy eigenvalue corresponding to the primitive state

$$E_g = c + 2 \sum_{\mu\nu} z_{\mu\nu} A_{\nu\mu}^* \tag{13}$$

where the parameters  $z_{\mu\nu}$  should satisfy the following algebraic equations:

$$A_{\mu\nu} + (\rho_\mu + \rho_\nu)z_{\mu\nu} + 4 \sum_{ij} A_{ij}^* z_{i\mu} z_{j\nu} = 0 \tag{14}$$

for  $1 \leq \mu, \nu \leq n$ . The possible roots  $\{z_{\mu\nu}\}$  of equation (14) should also keep the eigenvalue  $E_g$  real. It is obvious that there may be several sets of roots  $\{z_{\mu\nu}^{(p)}\}$  ( $p = 1, 2, \dots$ ), with which the eigenvalue  $E_g^{(v)}$  is real under some parametrizations depending on  $d_{ij}$  and  $B_{ij}$ . Therefore, generally there will be several different solutions to the problem.

Then, one-particle excitation states built on  $|g\rangle$  can easily be determined. First, write the one-particle excitation state up to a normalization factor as

$$|k = 1\rangle = \hat{F}|g\rangle \tag{15}$$

where

$$\hat{F} = \sum_{\mu} c_{\mu} b_{\mu}^+ \tag{16}$$

in which  $c_{\mu}$  ( $\mu = 1, 2, \dots, n$ ) are  $c$ -numbers to be determined. The eigenequation in this case is

$$\hat{H}\hat{F}|g\rangle = [\hat{H}, \hat{F}]|g\rangle + E_g\hat{F}|g\rangle = (E_1 + E_g)\hat{F}|g\rangle \tag{17}$$

where

$$[\hat{H}, \hat{F}] = \sum_{\mu} \rho_{\mu} c_{\mu} b_{\mu}^+ + 2 \sum_{\mu\nu} A_{\mu\nu}^* c_{\mu} b_{\nu}. \tag{18}$$

Using commutation relations

$$[b_i, T_{\mu\nu}^{(+)}] = \delta_{i\mu} b_{\nu}^+ + \delta_{i\nu} b_{\mu}^+ \tag{19}$$

it can be shown that

$$\begin{aligned} [\hat{H}, \hat{F}]|g\rangle &= \left( \sum_{\mu} \rho_{\mu} c_{\mu} b_{\mu}^+ + 2 \sum_{\mu\nu} A_{\mu\nu}^* c_{\mu} b_{\nu} \right) \Gamma_g|0\rangle \\ &= \sum_{\mu} \left( \rho_{\mu} c_{\mu} + 4 \sum_{ij} A_{ij}^* c_j z_{i\mu} \right) b_{\mu}^+ \Gamma_g|0\rangle. \end{aligned} \tag{20}$$

Combining equations (17) and (20), one finally obtains the following eigenequation for the eigenvalues  $E_1$  and the corresponding  $c$ -numbers  $\{c_{\mu}\}$ :

$$(\rho_{\mu} - E_1)c_{\mu} + 4 \sum_{ij} A_{ij}^* z_{i\mu} c_j = 0 \tag{21}$$

for  $\mu = 1, 2, \dots, n$ . It is clear that there are  $n$  eigenvalues of  $E_1$  determined by equation (21). Generally, there may exist some negative or complex eigenvalues  $E_1$ . However, complex and negative eigenvalues are physically unacceptable. A complex eigenvalue contradicts the fact that the Hamiltonian is Hermitian, while a negative eigenvalue will lead to an energy spectrum that is not lower bound. Therefore, the physically acceptable solutions of (21) are those with all  $n$  eigenvalues  $E_{1(v)}$  ( $v = 1, 2, \dots, n$ ) positive. One cannot select a part of positive eigenvalues from  $n$  eigenvalues  $\{E_{1(v)}\}$  as a solution to the problem because part of the eigenvector sets corresponding to the positive eigenvalues selected are incomplete. However, it is difficult to determine what conditions the coupling constants in (9) should satisfy in order to keep the solution within the lower bound spectrum. It can be seen from equation (21) that a necessary condition to keep the spectrum lower bound is

$$\sum_{\mu} \rho_{\mu} + 4 \sum_{ij} A_{ij}^* z_{ij} > 0. \tag{22}$$

But (22) is not sufficient. A trivial case is  $A_{\mu\nu} = 0$  for  $\{1 \leq \mu, \nu \leq n\}$ , in which  $\rho_{\mu} > 0$  with  $1 \leq \mu \leq n$  must be satisfied. If all eigenvalues  $E_{1(v)}$  ( $v = 1, 2, \dots, n$ ) are positive, one can prove that ‘ $k$ -particle’ excitation states up to a normalization factor can be written as

$$|k\rangle = \begin{cases} \sum_{\mu=0}^{[k/2]} \sum'_{i_1 i_2 \dots i_{2\mu}} a_{i_1 i_2 \dots i_{2\mu}}^{(2\mu)} \hat{F}_{i_1} \hat{F}_{i_2} \dots \hat{F}_{i_{2\mu}} |g\rangle & \text{for } k \text{ even} \\ \sum_{\mu=0}^{[k/2]} \sum'_{i_1 i_2 \dots i_{2\mu+1}} a_{i_1 i_2 \dots i_{2\mu+1}}^{(2\mu+1)} \hat{F}_{i_1} \hat{F}_{i_2} \dots \hat{F}_{i_{2\mu+1}} |g\rangle & \text{for } k \text{ odd} \end{cases} \tag{23}$$

where  $[x]$  denotes the integer part of  $x$ , the prime indicates that none of the pairs of indices  $i_p$  and  $i_q$  in the summation are the same with  $1 \leq i_1 < i_2 < \dots < i_{\nu} \leq k$  and with  $\nu \leq k$

$$\hat{F}_{i_q} \equiv \sum_{\mu} c_{\mu}^{(i_q)} b_{\mu}^+ \tag{24}$$

and the coefficients  $a_{i_1 i_2 \dots i_p}^{(p)}$  should satisfy the following recurrence relation:

$$a_{i_1 i_2 \dots i_{p-2}}^{(p-2)} = \frac{\sum'_{lm} a_{i_1 i_2 \dots i_{p-2} lm}^{(p)} c(l, m)}{E^{(k)} - E_g - \sum_{v=1}^{p-2} E_{i_v}}. \tag{25}$$

Again the prime in (25) indicates that  $l \neq m \neq i_1, i_2, \dots, i_{k-2}$ . We also have for the coefficients

$$c(m, l) = c(l, m) = [[\hat{H}, \hat{F}_m], \hat{F}_l] = \sum_{\mu\nu} A_{\mu\nu}^* c_{\mu}^{(m)} c_{\nu}^{(l)}. \tag{26}$$

One may start from  $a_{1 2 \dots k}^{(k)} = 1$ , and then use equation (25) to derive the other coefficients  $a_{i_1 i_2 \dots i_p}^{(p)}$  for  $p \leq k - 2$ .

It should be noted that the indices  $\{i_1, i_2, \dots, i_{\nu}\}$  in (23) are just labels for defining the expansion. For example, when  $k = 2$  there are only two terms in the expansion with

$$|k = 2\rangle = (a_0^{(0)} + a_{12}^{(2)} \hat{F}_1 \hat{F}_2) |g\rangle. \tag{27}$$

Though the labels  $i_1, i_2$  in  $a_{i_1 i_2}^{(2)}$  are different, the operators  $\hat{F}_1$  and  $\hat{F}_2$  can then be taken as the

same. When  $n = 3$  for example, (27) gives six possible eigenstates:

$$\begin{aligned}
 |k = 2, 1\rangle &= (a^{(0)}(1) + a_{12}^{(2)}(1)\hat{F}_1\hat{F}_1)|g\rangle \text{ with energy eigenvalue } E_g + 2E_1 \\
 |k = 2, 2\rangle &= (a^{(0)}(2) + a_{12}^{(2)}(2)\hat{F}_1\hat{F}_2)|g\rangle \text{ with energy eigenvalue } E_g + E_1 + E_2 \\
 |k = 2, 3\rangle &= (a^{(0)}(3) + a_{12}^{(2)}(3)\hat{F}_1\hat{F}_3)|g\rangle \text{ with energy eigenvalue } E_g + E_1 + E_3 \\
 |k = 2, 4\rangle &= (a^{(0)}(4) + a_{12}^{(2)}(4)\hat{F}_2\hat{F}_2)|g\rangle \text{ with energy eigenvalue } E_g + 2E_2 \\
 |k = 2, 5\rangle &= (a^{(0)}(5) + a_{12}^{(2)}(5)\hat{F}_2\hat{F}_3)|g\rangle \text{ with energy eigenvalue } E_g + E_2 + E_3 \\
 |k = 2, 6\rangle &= (a^{(0)}(6) + a_{12}^{(2)}(6)\hat{F}_3\hat{F}_3)|g\rangle \text{ with energy eigenvalue } E_g + 2E_3.
 \end{aligned} \tag{28}$$

When  $k = 4$ , there will be eight terms in the expansion with

$$\begin{aligned}
 |k = 4\rangle &= (a_0^{(0)} + a_{12}^{(2)}\hat{F}_1\hat{F}_2 + a_{13}^{(2)}\hat{F}_1\hat{F}_3 + a_{14}^{(2)}\hat{F}_1\hat{F}_4 \\
 &\quad + a_{23}^{(2)}\hat{F}_2\hat{F}_3 + a_{24}^{(2)}\hat{F}_2\hat{F}_4 + a_{34}^{(2)}\hat{F}_3\hat{F}_4 + a_{1234}^{(4)}\hat{F}_1\hat{F}_2\hat{F}_3\hat{F}_4)|g\rangle
 \end{aligned} \tag{29}$$

where  $\hat{F}_{i_r}$  and  $\hat{F}_{i_q}$  with  $r \neq q$  in (29) can also be the same. Each case corresponds to a different excitation state. It should be stated that the coefficients  $c_j^{(i_q)}$  in equations (24) and (26) should satisfy eigenequation (21). Therefore, the eigenenergies  $E_i$  for any  $i$  can only be taken as  $n$  values. By denoting these  $n$  eigenvalues of (21) as  $\{E_{\tau_i} \equiv E_{1(\tau_i)}\}$  ( $i = 1, 2, \dots, n$ ) it can easily be proven that the ' $k$ -particle' excitation energy can be rewritten as

$$E^{(k)} = E_g + \sum_i k_i E_{\tau_i} \tag{30}$$

with

$$\sum_{i=1}^n k_i = k \tag{31}$$

where  $[k_1, k_2, \dots, k_n]$  is an integer partition of  $k$ . Hence, the energy spectrum is still harmonic.

If there are several sets of solutions of equation (14), any one set of these solutions and the  $k$ -particle excitations built on (11) according to equation (21) form a complete set  $\{|k; z_{\mu\nu}\rangle\}$  with  $k = 0, 1, 2, \dots$ . These different sets of solutions cannot be the eigenstates of (9) simultaneously, otherwise the eigenstates will be over-complete because  $\langle k; z'_{\mu\nu} | k; z_{\mu\nu} \rangle \neq 0$ . In fact,  $\{|k; z_{\mu\nu}\rangle\}$  and  $\{|k; z'_{\mu\nu}\rangle\}$  are different sets of eigenstates spanning the same sub-Hilbert space. In order to illustrate this conclusion, let us consider a concrete example of Hamiltonian (2) for  $n = 2$  with  $c = 0$ ,  $d_{11} = 0.5$ ,  $d_{22} = 1.0$ ,  $d_{12} = d_{21} = 0.2$ ,  $B_{12} = B_{21} = 0.2$ ,  $B_{11} = B_{22} = 0$ , which are given in arbitrary units of energy. After transformation (8), we have  $\rho_1 = 1.008$ ,  $\rho_2 = 0.492$ ,  $A_{11} = -A_{22} = 0.193$ ,  $2A_{12} = -0.104$ . Using Mathematica, one can show that there are four sets of real solutions as follows.

*Case 1.*  $z_{11} = 0.2472$ ,  $z_{22} = -0.0998$ ,  $2z_{12} = 0.0768$ . The corresponding one-particle excitation energies calculated from equation (19) are all positive with  $E_{\tau_1} = 0.5673$ ,  $E_{\tau_2} = 1.1846$ . In this case, the primitive state (11) is the ground state. The corresponding ground state energy is 0.1177. Hence, the energy spectrum is lower bound.

*Case 2.*  $z_{11} = -0.0996$ ,  $z_{22} = 0.9990$ ,  $2z_{12} = 0.0979$ . The corresponding one-particle excitation energies calculated from (19) are  $E_{\tau_1} = 0.9236$ ,  $E_{\tau_2} = -0.2924$ . The primitive state energy is  $-0.4465$ .

*Case 3.*  $z_{11} = -2.4502$ ,  $z_{22} = 0.24435$ ,  $2z_{12} = 0.24027$ . The corresponding one-particle excitation energies calculated from (19) are  $E_{\tau_1} = -0.9236$ ,  $E_{\tau_2} = 0.2924$ . The primitive state energy is  $-1.0888$ .

For both cases 2 and 3, the energy spectrum is neither lower bound nor upper bound and consists of both positive and negative parts.

*Case 4.*  $z_{11} = -2.364$ ,  $z_{22} = 0.9542$ ,  $2z_{12} = 0.7351$ . The corresponding one-particle excitation energies are  $E_{\tau_1} = -0.935$ ,  $E_{\tau_2} = -0.2924$ . The primitive state energy is  $-1.4321$ . In this case, the energy spectrum is upper bound.

It is clear that these four cases are all the possible solutions. However, only a lower bound spectrum is acceptable in physical applications because frequencies of the quasi-particle excitations  $\omega_{\tau_i} = E_{\tau_i}/\hbar$  are positive. Therefore, only case 1 is the physical solutions to the problem. The situation will be more complicated for general  $n$  cases, but the conclusion for the  $n = 2$  case still applies for general  $n$ , i.e., the physical ground state is the VCS of  $Sp(2n, R) \supset U(N)$  built on the lowest weight state of  $U(n)$ .

In summary, the eigenvalue problem of the Hamiltonian built from a linear combination of all generators of  $Sp(2n, R)$ , which corresponds to  $n$ -coupled harmonic oscillators, is exactly solved using a simple algebraic procedure. It is found that, generally, there may be several sets of solutions. However, only lower bound solutions are acceptable in physical problems, in which the ground states of the Hamiltonian are the VCS of  $Sp(2n, R) \supset U(n)$  built on the lowest weight state of  $U(n)$ . The results show that the spectrum is determined by one-boson excitation energies built on the primitive state given by (11), and thus still remains harmonic.

### Acknowledgments

One of us (FP) is grateful to Professor J R Klauder for helpful discussions on the subject. This paper was supported by the US National Science Foundation through a regular grant, no 9970769, and a Cooperative Agreement, no EPS-9720652, that includes matching from the Louisiana Board of Regents Support Fund, and by the Science Foundation of the Liaoning Education Commission no 990311011.

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